

LECTURE NOTES ON STABILIZATION OF CONTACT OPEN BOOKS

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ABSTRACT. This note explains how to relate some contact geometric operations, such as surgery, to operations on an underlying contact open book. In particular, we shall give a simple proof of the fact that stabilizations of contact open books yield contactomorphic manifolds.

Let us remark that the results in this note are all well known to experts. This note just aims to provide some references for these results.

1. INTRODUCTION

The correspondence between open books and contact structures as established by Giroux [6] has been extremely fruitful in understanding contact structures both in dimension 3 and in higher dimensions.

In general, this correspondence looks as follows. Given a Weinstein manifold W and a symplectomorphism ψ of W that is the identity near ∂W , we can endow the mapping torus of (W, ψ) with a natural contact form. The boundary of this mapping torus is diffeomorphic to $\partial W \times S^1$, which allows us to glue in a copy of $\partial W \times D^2$. The latter set can be given a contact form which glues nicely to the one on the mapping torus.

Conversely, every compact coorientable contact manifold can in fact be obtained by this construction. However, supporting open books for contact manifolds are not unique. For instance, one has a stabilization procedure, which does not change the contact structure, but it does change the open book. Suppose we are given a contact open book $\text{Open}(W, \psi)$ with a Lagrangian disk L in a page W such that ∂L is a Legendrian sphere in ∂W . We obtain a new page \tilde{W} by attaching a symplectic handle to W along ∂L . The monodromy ψ can be extended as the identity on the symplectic handle. Since \tilde{W} contains a Lagrangian sphere formed by L and the core of the symplectic handle, we can compose the monodromy ψ with a right-handed Dehn twist τ_L along this Lagrangian sphere. This leads us to the (positive) stabilization of $\text{Open}(W, \psi)$, which is given by $\text{Open}(\tilde{W}, \tau_L \circ \psi)$. According to Giroux the stabilization is contactomorphic to the original contact manifold.

In dimension 3 the above correspondence is even better. Giroux has shown that on a compact, orientable 3-manifold M , open books for M up to (positive) stabilization correspond bijectively to isotopy classes of contact structures on M .

The goal of this note is to clarify some of these well-known notions and to provide proofs for some of them. We shall discuss the relation between contact surgery and open books: subcritical handle attachment along isotropic spheres in

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the binding can be seen as handle attachment to the page of the open book, whereas Legendrian surgery along a Legendrian sphere L in a page can be seen as composing the initial monodromy with a right-handed Dehn twist along L . This implies the well-known assertion that contact open books whose monodromy is isotopic to the product of right-handed Dehn twists are Stein fillable. We also provide a proof of the fact that stabilization does not change the contact structure.

Our proof is rather elementary and works almost entirely in the contact world. In particular, we shall not use Lefschetz fibrations (which could be used to look at the situation from another point of view): we basically interpret the handle attachment to the page and change of monodromy as successive contact surgeries which cancel each other. To see the latter though, we use symplectic handle cancellation.

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2. WEINSTEIN MANIFOLDS AND OPEN BOOKS

2.1. Weinstein and Stein. Let us first define the notion of Weinstein manifold.

Definition 2.1. Let M be a smooth manifold, and let $f : M \rightarrow \mathbb{R}$ be a smooth function. A vector field X on M is called **gradient-like** for the function f if $\mathcal{L}_X f > 0$ outside the critical points of f .

Definition 2.2. Let (W, ω) be a symplectic manifold. A proper function $f : W \rightarrow [0, \infty[$ is called ω -**convex** if it admits a complete gradient-like Liouville vector field X , i.e. $\mathcal{L}_X \omega = \omega$. We say (W, ω) is a **Weinstein manifold** if there exists an ω -convex Morse function.

Remark 2.3. From this definition it follows that all ends of a Weinstein manifold W are convex, i.e. they look like symplectizations. Indeed, let f be an ω -convex function and X be a complete gradient-like Liouville vector field for f . Since the vector field X is assumed to be complete, W cannot have any boundary components, because critical points of X must be isolated by the Morse condition.

Furthermore, since the Liouville vector field X is gradient-like for f , we see that X is positively transverse to regular level sets of f . Combined with properness of f this implies convexity of the ends.

Remark 2.4. We shall also apply the definition of ω -convex function to general symplectic cobordisms. In such a case the function f may not be bounded from below. The most basic example is a symplectization $(\mathbb{R} \times M, \omega = d(e^t \alpha))$, where the function $f(t, x) = e^t$ is ω -convex for $X = \frac{\partial}{\partial t}$.

Note that $i_X \omega$ defines a primitive of ω , so Weinstein manifolds are exact symplectic. For the sake of completeness, let us briefly recall some related notions.

Definition 2.5. Let (W, J) be an almost complex manifold. A function $f : W \rightarrow \mathbb{R}$ is said to be **strictly plurisubharmonic** if

$$g(X, Y) := -d(df \circ J)(X, JY)$$

for all vectors X, Y defines a Riemannian metric.

Let (W, J) be a Stein manifold. By a theorem of Grauert, (W, J) admits a strictly plurisubharmonic function f . Denote the associated symplectic form $-d(df \circ J)$ on

W by ω_f . We then see that strictly plurisubharmonic functions on Stein manifolds are examples of ω_f -convex Morse functions, i.e. Stein manifolds are Weinstein. Indeed, by solving the equation

$$i_X \omega_f = -df \circ J,$$

we obtain a Liouville vector field that is gradient-like for f , as $0 \geq \omega_f(X, JX) = df(X)$.

According to Eliashberg the converse is also true, but since we are only interested in the exact symplectic structure rather than the complex structure, we shall formulate everything using Weinstein manifolds.

Remark 2.6. A **compact Weinstein manifold** (Σ, ω) is a compact symplectic manifold with boundary K that can be embedded into a Weinstein manifold (W, ω) with an ω -convex function f such that Σ is given as the preimage $f^{-1}([0, C])$, and such that C is a regular level set of f . Note that such a regular level set is automatically contact.

2.2. Contact open books.

Definition 2.7. An **abstract (contact) open book** (Σ, λ, ψ) consists of a compact Weinstein manifold (Σ, λ) , and a symplectomorphism $\psi : \Sigma \rightarrow \Sigma$ with compact support such that $\psi^* d\lambda = d\lambda$.

Let us now show that an abstract contact open book corresponds to a contact manifold with a supporting open book.

By a lemma of Giroux [7] we can assume that $\psi^* \lambda = \lambda - dh$. We choose the function h to be positive. For completeness, here is the lemma and a proof.

Lemma 2.8 (Giroux). *The symplectomorphism ψ can be isotoped to a symplectomorphism $\hat{\psi}$ that is the identity near the boundary and that satisfies*

$$\hat{\psi}^* \lambda = \lambda - dh .$$

Proof. Let us denote the 1-form $\psi^* \lambda - \lambda$ by μ . Since $d\lambda$ is non-degenerate, we find a unique solution Y to the equation $i_Y d\lambda = -\mu$. The flow of the vector field Y preserves $d\lambda$, because μ is closed,

$$\mathcal{L}_Y d\lambda = d i_Y d\lambda = -d\mu = 0 .$$

Since ψ is the identity near the boundary, μ and hence Y vanishes near the boundary. If we denote the time- t flow of Y by φ_t , then we see that $\hat{\psi} = \psi \circ \varphi_1$ is a symplectomorphism that is the identity near the boundary. Note that $\mathcal{L}_Y \mu = 0$, so $\varphi_t^* \mu = \mu$ for all t . We check that the difference of the pullback of λ and λ is indeed exact. We have

$$(\psi \circ \varphi_1)^* \lambda - \lambda = \varphi_1^* (\mu + \lambda) - \lambda = \mu + \varphi_1^* \lambda - \lambda .$$

On the other hand, we can express the difference $\varphi_1^* \lambda - \lambda$ as

$$\begin{aligned} \varphi_1^* \lambda - \lambda &= \int_0^1 \frac{d}{dt} \varphi_t^* \lambda dt = \int_0^1 (\varphi_t^* \mathcal{L}_Y \lambda) dt = \int_0^1 \varphi_t^* (i_Y d\lambda + d(i_Y \lambda)) dt \\ &= -\mu + d \int_0^1 \varphi_t^* (i_Y \lambda) dt . \end{aligned}$$

Moving μ to the left-hand-side, we see that $\mu + \varphi_1^* \lambda - \lambda$ is exact, which shows the claim of the lemma. \square

Now we can define

$$A_{(\Sigma, \psi)} := \Sigma \times \mathbb{R} / (x, \varphi) \sim (\psi(x), \varphi + h(x)) .$$

This mapping torus carries the contact form

$$\alpha = \lambda + d\varphi .$$

Since ψ is the identity near the boundary of Σ , a neighborhood of the boundary looks like

$$(-\frac{1}{2}, 0] \times \partial\Sigma \times S^1 ,$$

with contact form

$$\alpha = e^r \lambda|_{\partial\Sigma} + d\varphi .$$

Denote the annulus $\{z \in \mathbb{C} \mid r < |z| < R\}$ by $A(r, R)$. We can glue the mapping torus $A_{(\Sigma, \psi)}$ along its boundary to

$$B_\Sigma := \partial\Sigma \times D^2$$

using the map

$$\begin{aligned} \Phi_{\text{glue}} : \partial\Sigma \times A(1/2, 1) &\longrightarrow (-1/2, 0] \times \partial\Sigma \times S^1 \\ (x; re^{i\varphi}) &\longmapsto (1/2 - r, x, \varphi) . \end{aligned}$$

Pulling back the form α by Φ_{glue} , we obtain

$$e^{1/2-r} \lambda|_{\partial\Sigma} + d\varphi$$

on $\Sigma \times A(1/2, 1)$, which can be easily extended to a contact form

$$\beta = h_1(r) \lambda|_{\partial\Sigma} + h_2(r) d\varphi$$

on the interior of B_Σ by requiring that h_1 and h_2 are functions from $[0, 1)$ to \mathbb{R} whose behavior is indicated in Figure 1; $h_1(r)$ should have exponential drop-off and $h_2(r)$ should quadratically increase near 0 and be constant near 1.

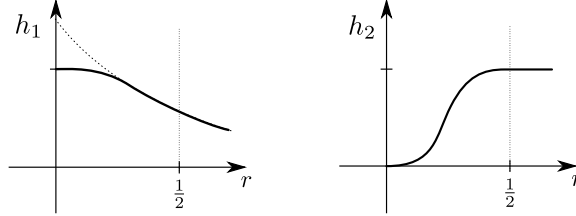


FIGURE 1. Functions for the contact form near the binding

The union $M := A_{(\Sigma, \psi)} \cup_{\partial} B_\Sigma$ is called an **abstract open book** for M . Note that the contact forms α on $A_{(\Sigma, \psi)}$ and β on B_Σ glue together to a globally defined contact form.

We shall call the resulting contact manifold, which we denote by $\text{Open}(\Sigma, \lambda; \psi)$, a **contact open book**. We shall sometimes drop the primitive λ of the symplectic form in our later notation.

Remark 2.9. Note that $\text{Open}(\Sigma, \lambda; \psi)$ has the structure of a fibration over S^1 away from the set B . Hence we can talk about the monodromy of an open book, which can be obtained by lifting the tangent vector field to S^1 , given by ∂_φ , to a vector field on A . If we rescale the function h to 2π then the time- 2π flow gives the monodromy. Note that a positive function times the Reeb vector field is a suitable lift of ∂_φ . As a result, we see that the monodromy is given by ψ^{-1} .

We should also point out that there are various conventions in use at this point. Some papers refer to ψ as the monodromy and Milnor [9] used the word *characteristic homeomorphism*.

Definition 2.10. An **open book** on M is a pair (K, ϑ) , where

- K is a codimension 2 submanifold of M with trivial normal bundle, and
- $\vartheta : M - B \rightarrow S^1$ endows $M - K$ with the structure of a fiber bundle over S^1 such that ϑ gives the angular coordinate of the D^2 -factor of a neighborhood $B \times D^2$ of B .

The set K is called the **binding** of the open book. A fiber of ϑ together with the binding is called a **page** of the open book.

Remark 2.11. The typical situation of an open book is the following. Let K be a knot in a 3-manifold. For special knots, so-called fibered knots, the complement fibers over S^1 in a nice way: this is equivalent to an open book. A well-known example is the unknot in S^3 .

In order to define the notion of adapted open book, we need to discuss the orientations involved. Suppose M is an oriented manifold with an open book (K, ϑ) . Since we regard S^1 as an oriented manifold, each page Σ gets an induced orientation such that the orientation of $M - K$ as a bundle over S^1 matches the one coming from M . If this orientation of the page Σ matches the orientation as a symplectic manifold, we call a symplectic form ω on Σ **positive**. We shall orient the binding K as the boundary of a page Σ using the outward normal. If, on the other hand, this orientation matches the one coming from a contact form α , i.e. $\alpha \wedge d\alpha^n$, then we say that α induces a **positive contact structure**.

Definition 2.12. A positive contact structure ξ on an oriented manifold M is said to be **carried by an open book** (B, ϑ) if ξ admits a defining contact form α satisfying the following conditions.

- α induces a positive contact structure on B , and
- $d\alpha$ induces a positive symplectic structure on each fiber of ϑ .

A contact form α satisfying these conditions is said to be **adapted** to (B, ϑ) .

Lemma 2.13. Suppose that B is a connected contact submanifold of a contact manifold (M, ξ) . A contact form α for (M, ξ) is adapted to an open book (B, ϑ) if and only if the Reeb field R_α of α is positively transverse to the fibers of ϑ , i.e. $R_\alpha(\vartheta) > 0$.

Proof. If $d\alpha$ is positive on each fiber of ϑ , then we can find tangent vectors v_1, \dots, v_{2n} to the page at a point x such that $i_{v_1 \wedge \dots \wedge v_{2n}} d\alpha^n > 0$. Hence

$$\iota_{R \wedge v_1 \wedge \dots \wedge v_{2n}} \alpha \wedge d\alpha^n > 0.$$

Since the pages and the S^1 direction also orient the manifold, we see that the Reeb field is positively transverse to the pages.

Conversely, if R_α is positively transverse to the fibers of ϑ , then $i_{R_\alpha} \alpha \wedge d\alpha^n > 0$, so in particular $d\alpha$ is a positive symplectic form on each fiber.

We assume B to be a contact submanifold, so we only need to check positivity. Note that

$$\int_{\partial P} \alpha \wedge d\alpha^{n-1} = \int_P d\alpha \wedge d\alpha^{n-1} = \int_P d\alpha^n > 0 .$$

Since the binding B was assumed to be connected, we see that (B, ϑ) is a supporting open book. \square

Proposition 2.14. *An abstract contact open book $\text{Open}(\Sigma, \psi)$ admits a natural open book carrying the contact structure ξ in the above construction.*

Proof. We define the binding of the abstract contact open book $\text{Open}(\Sigma, \psi)$ to be the submanifold $\partial\Sigma \times \{0\}$. The map ϑ from $M - B$ to S^1 can be defined by putting $\vartheta(x) = \varphi$ if $x = (p; r, \varphi)$ is a point in $\partial\Sigma \times D^2$. For points in A , we use the fact the A is a fiber bundle over S^1 . Moreover, the definitions coincide on the overlap of A and $\partial\Sigma \times D^2$.

The Reeb field of the abstract contact open book $\text{Open}(\Sigma, \lambda; \psi)$ as given by the above construction is ∂_φ , so it is positively transverse to all pages. This implies that the open book carries the associated contact structure. \square

2.3. Basic properties of open books.

2.3.1. Order and monodromy. In general, the resulting contact manifold depends on the monodromy, but there are some symmetries. For instance, if Σ is a convex symplectic manifold and ψ_1 and ψ_2 are symplectomorphisms, then

$$\text{Open}(\Sigma, \psi_2^{-1} \circ \psi_1 \circ \psi_2) \cong \text{Open}(\Sigma, \psi_1) .$$

Indeed, we can simply regard the mapping torus of the open book as three products $\Sigma \times I$ glued together. Gluing them in another order gives the same result.

This observation also implies the cyclic symmetry property,

$$\text{Open}(\Sigma, \psi_1 \circ \psi_2) \cong \text{Open}(\Sigma, \psi_2 \circ \psi_1) .$$

Indeed, if we conjugate $\psi_2 \circ \psi_1$ by ψ_2 , we get the above expression.

2.4. Important examples of monodromies. In general, the group of symplectomorphisms on a symplectic manifold is poorly understood. In fact, in many cases, such (D^6, ω_0) , it is unknown whether every symplectomorphism is isotopic to the identity (relative to the boundary).

There is, however, one way to construct candidates of symplectomorphisms that are in general not isotopic to the identity. Suppose that (W, ω) is a symplectic manifold with an embedded Lagrangian sphere $L \subset W$. By the Weinstein neighborhood theorem, a neighborhood $\nu_W(L)$ is symplectomorphic to the canonical symplectic structure on $(T^*S^n, d\lambda_{can})$.

Hence we consider the symplectic manifold $(T^*S^n, d\lambda_{can})$, where λ_{can} is the canonical 1-form. In local coordinates, this form is given by $\lambda_{can} = pdq$. To describe a so-called Dehn twist, we first regard this manifold as a submanifold of \mathbb{R}^{2n+2} by using coordinates

$$(q, p) \in \mathbb{R}^{2n+2}$$

subject to the following relations

$$(2.1) \quad q \cdot q = 1 \text{ and } q \cdot p = 0 .$$

With these coordinates the canonical 1-form λ_{can} on T^*S^n is given by

$$\lambda_{can} = p dq.$$

Define an auxiliary map describing the normalized geodesic flow

$$\sigma_t(q, p) = \begin{pmatrix} \cos t & |p|^{-1} \sin t \\ -|p| \sin t & \cos t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$

Then define

$$\tau(q, p) = \begin{cases} \sigma_{g_1(|p|)}(q, p) & \text{if } p \neq 0 \\ -\text{Id} & \text{if } p = 0. \end{cases}$$

Here g_1 is a smooth function with the following properties.

- $g_1(0) = \pi$ and $g_1'(0) < 0$.
- Fix $p_0 > 0$. The function $g_1(|p|)$ decreases to 0 at p_0 after which it is identically 0.

Note that the conditions imply that τ is actually a smooth map. See Figure 2. The map τ is called a **(generalized) right-handed Dehn twist**.

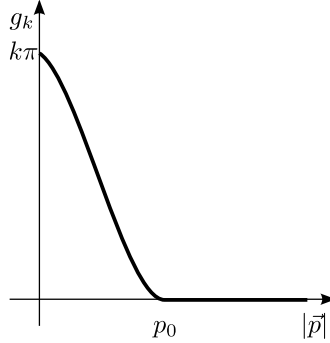


FIGURE 2. The amount of geodesic flow for a k -fold Dehn twist

Since τ is the identity near the boundary of T^*S^n , we can extend τ to a symplectomorphism of (W, ω) : simply extend τ to be the identity outside the support of τ .

3. CONTACT SURGERY AND SYMPLECTIC HANDLE ATTACHMENT

Let (M, ξ) be a contact manifold, and let S in M be an isotropic k -sphere with a trivialization ε of its conformal symplectic normal bundle. Then we can perform contact surgery along (S, ε) . We shall write the surgered contact manifold as

$$\widetilde{(M, \xi)}_{S, \varepsilon}$$

In case of Legendrian surgery, there is no choice for the framing ε , and consequently, we shall drop the framing from the notation in that case.

We shall now describe a model for contact surgery in terms of symplectic handle attachment. For later computations, we slightly modify Weinstein's original construction, [10].

3.1. "Flat" Weinstein model for contact surgery. Here we shall discuss a slightly modified version of the Weinstein model for contact surgery. Let $(M, \xi = \ker \alpha)$ be a contact manifold and suppose that S is an isotropic k -sphere in (M, ξ) with trivial conformal symplectic normal bundle, trivialized by ε . Using this framing ε and a neighborhood theorem, see Theorem 6.2.2 in [5], we can find a *strict* contactomorphism

$$\psi : (\nu(S), \alpha) \longrightarrow \left(\mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k-1)}, dz + p dq + \frac{1}{2}(x dy - y dx) \right),$$

where we regard $\mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k-1)}$ as a neighborhood of $\{0\} \times S^k \times \{0\}$.

Remark 3.1. We should point out that the contactomorphism ψ depends on the trivialization ε . As a result, the entire construction we shall describe now, depends on this choice. Note that this is unavoidable, since even smoothly the result of surgery depends on the choice of framing.

A priori, we can only expect a small neighborhood of S to be contactomorphic to a *small* subset of $\mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k)}$ via a strict contactomorphism, but we can enlarge this neighborhood by composing with the following non-strict contactomorphism

$$\begin{aligned} \varphi_C : \mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k)} &\longrightarrow \mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k)} \\ (z, q, p; x, y) &\longmapsto (Cz, q, Cp; \sqrt{C}x, \sqrt{C}y). \end{aligned}$$

Now consider the following model for contact surgery and symplectic handle attachment. Consider the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$. We shall use coordinates (x, y, z, w) , where there are $n - k - 1$ pairs of (x, y) coordinates and $k + 1$ pairs of (z, w) coordinates. The symplectic form is then given by

$$\omega_0 = dx \wedge dy + dz \wedge dw.$$

Note that the vector field

$$X = \frac{1}{2}(x\partial_x + y\partial_y) + 2z\partial_z - w\partial_w$$

is Liouville for ω_0 .

Now consider the set

$$S_{-1} := \{(x, y, z, w) \in \mathbb{R}^{2n} \mid |w|^2 = 1\}.$$

The Liouville field X is transverse to this set, and induces the contact form

$$\alpha = \frac{1}{2}(x dy - y dx) + 2z dw + w dz.$$

We see that the sphere $\{(x, y, z, w) \mid x = y = 0, z = 0, |w|^2 = 1\} \cong S^k$ describes an isotropic sphere in S_{-1} with trivial conformal symplectic normal bundle. We shall think of S_{-1} as a neighborhood of the isotropic sphere S , in other words S_{-1} can be thought of as the situation before surgery. In fact, the set S_{-1} is a standard neighborhood of an isotropic sphere of dimension k with trivial normal bundle, since we have the following contactomorphism,

$$\begin{aligned} \psi_W : \mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k-1)} &\longrightarrow S_{-1} \\ (z, q, p, x, y) &\longmapsto (x, y; zq + p, q). \end{aligned}$$

Here we regard the cotangent bundle T^*S^k as a subspace of $\mathbb{R}^{2(k+1)}$ by using coordinates $(q, p) \in \mathbb{R}^{2(k+1)}$, where $q^2 = 1$ and $q \cdot p = 0$. Note that ψ_W is a strict contactomorphism,

$$\begin{aligned} \psi_W^* \left(\frac{1}{2} (x dy - y dx) + 2z dw, dw + w dz \right) &= \\ \frac{1}{2} (x dy - y dx) + 2(zq + p)dq + q d(zq) + q dp &= \frac{1}{2} (x dy - y dx) + p dq + dz. \end{aligned}$$

To see that the latter step holds, use that $q dq = 0$ and $p dq + q dp = 0$.

We can combine the above three maps to obtain a contactomorphism from $\nu(S) \subset M$ to S_{-1} in the Weinstein model

$$(3.1) \quad \Phi_C := \psi_W \circ \varphi_C \circ \psi : \nu(S) \longrightarrow S_{-1}.$$

This map is not a strict contactomorphism, but since it multiplies the contact form with a constant rather than an arbitrary function, we can adapt the following lemma from [5], Lemma 5.2.4, for a gluing construction.

Lemma 3.2. *For $i = 0, 1$, let (M_i, α_i) be a (not necessarily closed) contact type hypersurface in a symplectic manifold (W_i, ω_i) with respect to the Liouville vector field Y_i . Suppose $\varphi : (M_0, \alpha_0) \rightarrow (M_1, \alpha_1)$ is a contactomorphism such that $\varphi^* \alpha_1 = C \alpha_0$ for some constant C . Then φ extends to a symplectomorphism between neighborhoods of M_0 and M_1 by sending flow lines of Y_0 to flow lines of Y_1 .*

Furthermore, we can choose a large C in Formula (3.1), which means that we can get arbitrary large neighborhoods in the Weinstein model.

Remark 3.3. We can also adapt the proof of Proposition 3.1 in [1] to obtain a contactomorphism from $\nu(S)$ to the full Weinstein model, i.e. a surjective map to S_{-1} . This contactomorphism is in general not strict, or even admissible for Lemma 3.2. Therefore we shall restrict ourselves to a contactomorphism as in Formula (3.1).

3.1.1. Attaching a symplectic handle. Let us begin by defining a symplectic handle. The contactomorphism Φ_C identifies the neighborhood $\nu(S) \subset M$ with a neighborhood of the isotropic sphere in S_{-1} . Suppose that the neighborhood provided by ψ has size

$$\text{size}_\psi(\nu(S)) := \max_{(x,y,z,w) \in \psi(\nu(S))} \sqrt{x^2 + y^2 + z^2} = \tilde{\varepsilon}.$$

Then by choosing $C > 2/\tilde{\varepsilon}$ we can ensure that the neighborhood provided by Φ_C has size larger than 1, i.e. the maximal (x, y, z) coordinates are larger than 1.

We first define the profile for the handle. Fix a small $\delta > 0$: this parameter serves as a smoothing parameter. Choose smooth functions $f, g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that

- f is increasing.
- $f(w) = 1$ for $w \in [0, 1 - \delta]$, $f(w) = w + \delta$ for $w > 1 - \delta/2$.
- g is increasing.
- $g(z) = z$ for $z < 1$, $g(w) = 1 + \delta$ for $w > 1 + \delta$.

See Figure 3 for a sketch of these functions.

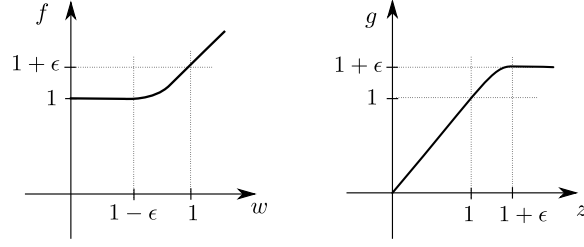


FIGURE 3. Functions for the profile of a symplectic handle

Define

$$F(x, y, z, w) := -f(w^2) + g(x^2 + y^2 + z^2).$$

Define a hypersurface $S_1 := \{(x, y, z, w) \mid F(x, y, z, w) = 0\}$. This hypersurface is of contact type, because the Liouville vector field X is transverse to S_1 ,

$$X(F) = \left(\frac{1}{2}(x^2 + y^2) + 2z^2 \right) g' + w^2 f' > 0$$

for points x, y, z, w such that $F(x, y, z, w) = 0$, as points with $g' = 0$ are precisely those with $w^2 = 1$, and points with $f' = 0$ are those with $x^2 + y^2 + z^2 = 1$. The hypersurface S_1 is meant to describe the result of the surgery along S . See Figure 4 for a sketch of the situation.

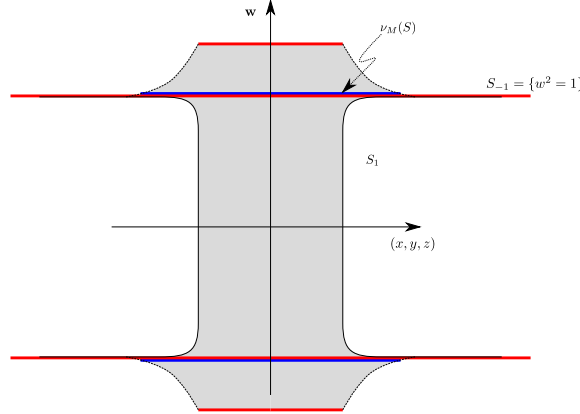


FIGURE 4. A symplectic handle in the flat Weinstein model

Remark 3.4. Instead of a profile for a symplectic handle described by the above function F , one more commonly chooses a profile of the form

$$x^2 + y^2 + z^2 - w^2 = c.$$

The advantage is that

$$G = x^2 + y^2 + z^2 - w^2$$

defines an ω -convex Morse function with respect to the Liouville field X with one critical point on the handle. The main reason for preferring F is that it simplifies

later computations. Note that topologically the two profiles are the same. Furthermore, one can adapt the ω -convex function G to the above profile as well. See the summary in Proposition 3.7.

In order to describe the surgery, we shall use handle attachment along a symplectic manifold (W, ω) with contact type boundary M .

Define the symplectic handle (H_{k+1}, ω_0) as follows. H_{k+1} consists of those points $p \in (\mathbb{R}^{2n}, \omega_0)$ such that

- There is a $t \in [0, 1]$ such that the time- t flow of X satisfies $Fl_t^X(p) \in \Phi_C(\nu(S))$. This is the gluing part of the symplectic handle.
- There are $t_1 \leq 0$ such that $Fl_{t_1}^X(p) \in \Phi_C(\nu(S))$ and $t_2 \geq 0$ such that $Fl_{t_2}^X(p) \in S_1$.

Let us now attach this symplectic handle H_{k+1} to (W, ω) . A neighborhood of the boundary of (W, ω) is symplectomorphic to $([-1, 0] \times M, d(e^t, \alpha))$; call this symplectomorphism $\psi_\partial : \nu_W(M) \rightarrow [-1, 0] \times M$ (note that we can attach a piece of a symplectization of M to W to ensure we have such a neighborhood). In particular, we have a symplectomorphism

$$\psi_\partial : \nu_W(\nu_M(S)) \longrightarrow [-1, 0] \times \nu_M(S).$$

We can compose this symplectomorphism with the map

$$\begin{aligned} \tilde{\Phi}_C : [-1, 0] \times \nu_M(S) &\longrightarrow H_1 \\ (t, p) &\longmapsto Fl_t^X(\Phi_C(p)). \end{aligned}$$

This map is also a symplectomorphism, cf. Lemma 3.2 (or rescale the symplectic form on H_{k+1}).

Now attach the symplectic handle

$$\tilde{W} := W \cup H_{k+1} / \sim.$$

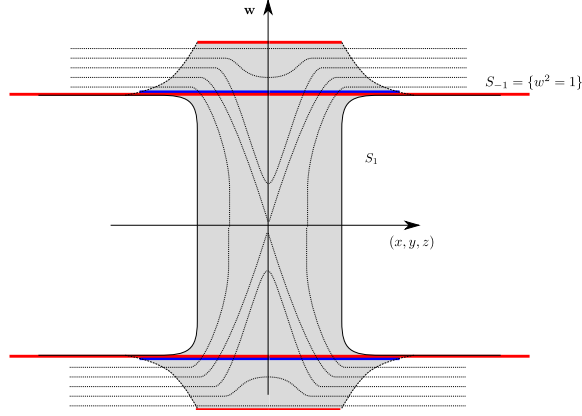
Here we glue x in $\nu_W(\nu_M(S)) \subset W$ to y in H_1 if and only if $\tilde{\Phi}_C \circ \psi_\partial(x) = y$. By Lemma 3.2 the resulting manifold \tilde{W} is again symplectic and its boundary is a contact manifold that is diffeomorphic to the surgered manifold $(\tilde{M}, \xi)_{S, \varepsilon}$, obtained by performing surgery on M along the isotropic submanifold S with framing ε .

Definition 3.5. The above attaching procedure is called **symplectic handle attachment** along S at the convex end of W . We call the attachment **subcritical** if $\dim S < n$ and **critical** if $\dim S = n$. The induced operation on the convex end is called **contact surgery** along S . The contact surgery is called **subcritical** if $\dim S < n$ and **critical** or **Legendrian** if $\dim S = n$.

Remark 3.6. Since we attach a symplectic handle to a cobordism by gluing flow lines of the respective Liouville fields, we see that we can extend the Liouville field defined in a neighborhood of the convex end of W to the new symplectic manifold $(\tilde{W}, \tilde{\omega})$.

As alluded to in Remark 3.4, there is a function on a symplectic handle that is ω_0 -convex for the Liouville field X . By slightly modifying the ω -convex function f on W , we can glue this function to the one on the symplectic handle. This is sketched in Figure 5

Let us summarize the above discussion in the following proposition.

FIGURE 5. A sketch of the modifications to glue ω -convex functions

Proposition 3.7. *Let (W, ω) be a symplectic cobordism. Suppose that $i : S \rightarrow \partial W$ is an embedded isotropic k -sphere in the convex end of W whose conformal symplectic normal bundle is trivialized by ε .*

Then we can attach a handle H_1 to W along S with framing ε to obtain a symplectic cobordism $(\tilde{W}, \tilde{\omega})$.

Furthermore, if (W, ω) admits an ω -convex function f , then f can be extended to an $\tilde{\omega}$ -convex function \tilde{f} on \tilde{W} such that \tilde{f} has only one additional critical point.

Remark 3.8. We see that we can attach symplectic handles under rather mild assumptions to the convex end of a symplectic manifold. The converse, i.e. attaching handles to the concave end of a symplectic manifold is much more restrictive. Indeed, there are many examples of non-fillable contact manifolds, which illustrates that concave handle attachment has additional requirements.

3.2. Symplectic handle cancellation. The main technical tool we shall use is Lemma 3.6b from [3]. Here is a formulation that is suitable for our purposes.

Lemma 3.9 (Eliashberg). *Let (W, ω) be a symplectic manifold and f be an ω -convex function.*

Let p and q be non-degenerate critical points of f and $d \in]f(p), f(q)[$. Suppose that

- $\text{index}_q(f) = \text{index}_p(f) + 1$.
- *The sphere S_q^- , obtained by intersecting the stable manifold $W^s(q)$ with the level set $\{x \mid f(x) = d\}$, intersects the sphere S_p^+ , formed by intersecting the unstable manifold $W^u(p)$ with $\{x \mid f(x) = d\}$, transversely in one point.*

Then the critical points can be cancelled by a J -convex deformation of f in a neighborhood of $[f(p), f(q)]$.

A very similar statement with proof can also be found in [2], Proposition 10.9.

Given the lemma, we can perform symplectic handle cancellation in a way similar to the one in the smooth case, see [8], Theorem 5.6. We shall briefly describe the particular setup which we shall use. This will be the simplest case of handle cancellation: it can occur after consecutive attachment of handles with index difference 1.

Let (W_1^{2n}, ω) be a symplectic manifold such that $M_1 \subset \partial W_1$ is a convex end. Choose an ω -convex function f_1 near the convex end and let X_1 be the associated Liouville vector field.

Now suppose that $S_1 \subset M_1$ is an isotropic $(n-2)$ -sphere with a trivialization ε of its conformal symplectic normal bundle. Suppose furthermore that S_1 bounds a Legendrian $(n-1)$ -disk D_1 in M_1 . Now form the symplectic manifold (W_2, ω_2) by attaching a symplectic $(n-1)$ -handle along S_1 ,

$$(W_2, \omega_2) = W_1 \cup_{S_1, \varepsilon} H_{n-1}.$$

The ω_1 -convex f_1 can be extended to an ω_2 -convex function f_2 as mentioned in Remark 3.6: this new ω_2 -convex function has one additional critical point, corresponding to the middle of the handle. We shall denote this critical point by p .

Note that the convex end M_1 is surgered into a new convex end $M_2 \subset \partial W_2$. This convex end comes with a Legendrian $(n-1)$ -sphere S_2 which is formed as follows.

First observe that there is a parallel copy D_2 of the core of H_{n-1} which is a Legendrian $(n-1)$ -disk. More explicitly, we can think of H_{n-1} as $D^2 \times T^*S^{n-1}$. Then put

$$\begin{aligned} \varphi: D^{n-1} &\longrightarrow H_{n-1} = D^2 \times T^*D^{n-1} \\ w &\longmapsto (h_1(|w|)x_0, 0, h_2(|w|)w). \end{aligned}$$

Here $x_0 = (1, 0) \in D^2$, and h_1 and h_2 are functions parametrizing the profile S_1 . We see directly that $\alpha = 1/2(x dy - y dx) + 2z dw + w dz$ restricts to 0 on $D_2 := \varphi(D^{n-1})$. After this, we can match D_1 (which is partially removed after the handle attachment of H_{n-1}) to glue to D_2 . This gives the Legendrian sphere S_2 .

Remark 3.10. To visualize the handle cancellation that is going to occur in the next step, observe that S_2 intersects the belt sphere of H_{n-1} transversely in one point, namely in $\varphi(0)$.

Since S_2 is Legendrian, the conformal symplectic normal bundle is trivial, so we can form W_3 by critical n -handle attachment along S_2 without reference to a framing,

$$(W_3, \omega_3) := W_2 \cup_{S_2} H_n.$$

As before, we can extend the ω_2 -convex function f_2 to an ω_3 -convex function f on W_3 . Denote the additional critical point of f_3 by q . We shall denote the gradient-like Liouville vector field on W_3 by X_3 . The convex end M_2 is surgered yielding the contact manifold M_3 .

Now intersect a level set $\{f_3 = d\}$, with d between $f_3(p)$ and $f_3(q)$, with the stable manifold $W^s(q)$ and the unstable manifold $W^u(p)$ to form the spheres $S^-(q)$ and $S^+(p)$, respectively. These spheres intersect transversely in one point, as we can see from the unique flow line of the Liouville vector field X_3 from p to q .

This means that Lemma 3.9 applies, so we can deform f_3 to another ω_3 -convex function g_3 such that $g_3(x) = f_3(x)$ on sublevel sets $\{f_3 < c = f(p) - \delta\}$. In particular, on such sublevel sets g_3 coincides with f_1 . Furthermore, g_3 has no critical points whenever $g_3(x) \geq c$. This means that $\{g_3(x) \geq c\}$ looks like a symplectization, so we conclude that the completion of W_1 , i.e. the manifold obtained from W_1 by attaching the positive end of a symplectization, is symplectomorphic to the completion of W_3 .

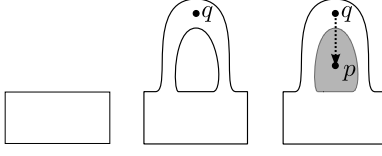


FIGURE 6. Cancellation of symplectic handles: the creation of a unique flow line between the critical points q and p

We summarize the conclusion in the following lemma.

Lemma 3.11 (Handle cancellation in successive handle attachment). *Let (W_1, ω_1) and (W_3, ω_3) be the symplectic manifolds as formed above by successive handle attachment. Then the completion of (W_1, ω_1) is symplectomorphic to the completion of (W_3, ω_3) . In particular, M_1 is contactomorphic to M_3 .*

4. SURGERY AND OPEN BOOKS

In this section we try to describe some relations between contact surgery and open books. Let us summarize the results that will be proved below. If an isotropic sphere S lies in the binding of an open book and if the framing is compatible with the open book, then subcritical surgery along S can also be described in terms of handle attachment to the pages of an open book.

On the other hand, critical contact surgery can be regarded as a change in the monodromy of the open book, if the sphere used for the surgery lies nicely in a page.

We apply this to show that stabilization of open books leads to contactomorphic contact manifolds. The basic strategy is the following. To stabilize an open book we attach an n -handle to the $2n$ -dimensional page forming a new Lagrangian sphere and change the monodromy by composing with a right-handed Dehn twist along the newly formed Lagrangian sphere.

We shall show that the handle attachment to the page can be realized by a subcritical handle attachment to the convex end of $[-1, 0] \times M$ and that the change in monodromy is realized by a critical handle attachment. The latter turns out to cancel the former, so we obtain the same contact manifold.

4.1. Subcritical surgery and open books. Let us first describe the situation for trivial monodromy, since that situation is more easily visualizable. Let Σ be a compact Stein manifold with boundary $B := \partial\Sigma$ and consider the open book

$$M := \text{Open}(\Sigma, \text{Id}) .$$

Suppose that S is an isotropic (possibly Legendrian) sphere in B with a trivialization ε of its conformal symplectic normal bundle. We can perform contact surgery along (S, ε) giving rise to a contact manifold \hat{B} . The associated surgery cobordism also gives a Stein filling for \hat{B} , which we will denote by $\hat{\Sigma}$. Alternatively, $\hat{\Sigma}$ can be regarded as the Stein manifold obtained from Σ by handle attachment along (S, ε) .

Note that S also gives rise to an isotropic submanifold of M . Indeed, we have an isotropic sphere in the binding: take $S_M := S \times \{0\} \subset B \times D^2$. Since we have the following contact form near the binding,

$$\lambda + x dy - y dx ,$$

we see that we also get a trivialization of the conformal symplectic normal bundle of $S_M \subset M$, given by $\varepsilon_M := \varepsilon \oplus \langle \partial_x, \partial_y \rangle$. For later use, it is useful to give the last factor a name,

$$\varepsilon_{D^2} = \langle \partial_x, \partial_y \rangle .$$

Contact surgery on M along (S_M, ε_M) gives the subcritical fillable contact manifold $\widetilde{M} := \partial(\Sigma \times D^2)$, as we can see from performing handle attachment to the filling $\Sigma \times D^2$ of M . On the level of open books, we see that \widetilde{M} has a supporting open book with page $\widetilde{\Sigma}$ and the identity as monodromy.

This setup also describes the general situation, since the surgery takes place near the binding. As a result, we have the following proposition,

Proposition 4.1. *Let Σ be a compact Stein manifold with boundary B and let ψ be a compactly supported symplectomorphism such that*

$$M := \text{Open}(\Sigma, \psi)$$

is a contact open book. Suppose that S_B is an isotropic $(k-1)$ -sphere in the binding B with a trivialization ε of its conformal symplectic normal bundle in B . Then there is a corresponding isotropic $(k-1)$ -sphere $S_M \subset M$ with trivialization $\varepsilon \oplus \varepsilon_{D^2}$ of its conformal symplectic normal bundle such that

$$\text{Open}(\Sigma \cup_{S_B} k\text{-handle}, \psi \cup_{S_B} \text{Id}) \cong \widetilde{\text{Open}(\Sigma, \psi)}_{(S_M, \varepsilon \oplus \varepsilon_{D^2})} .$$

In other words, this kind of subcritical surgery is realized by handle attachment to the page of an open book without changing the monodromy.

4.2. Critical surgery and open books. Now consider a contact open book $M := \text{Open}(\Sigma, \psi)$ having a Lagrangian sphere L_S in a page. We can assume that L_S represents a Legendrian sphere in $\Sigma \times \mathbb{R}$, or in other words in the contact open book M .

Lemma 4.2. *Let $M^{2n+1} = \text{Open}(\Sigma, \psi)$ be a contact manifold of dimension greater than 3. If L_S is a Lagrangian sphere in the page of a contact open book M^{2n+1} , then we can isotope the contact structure on M^{2n+1} and find a supporting open book with symplectomorphic page and isotopic monodromy such that L_S becomes Legendrian in M^{2n+1} .*

Proof. Suppose λ is a primitive of the symplectic form ω on $\widetilde{\Sigma}$. Then on a Weinstein neighborhood of L_S we can find a primitive

$$\lambda_{\text{can}} = p dq$$

of the symplectic form ω , where (q, p) are coordinates on $T^*S^n \cong \nu(L_S)$. Since $\lambda - \lambda_{\text{can}}$ is closed, we can find a function g such that

$$\lambda - \lambda_{\text{can}} = dg ,$$

because $H_{dR}^1(S^n) = 0$ as $n > 1$. Now put

$$\widetilde{\lambda} := \lambda - d(\rho g) ,$$

where ρ is a smooth cut-off function that is 1 in a neighborhood of L_S with support in the Weinstein neighborhood $\nu(L_S)$. Note that $d\widetilde{\lambda} = d\lambda$ is still symplectic.

On the Lagrangian sphere L_S , $\widetilde{\lambda}$ vanishes, so L_S lies in the kernel of the contact form $dt + \widetilde{\lambda}$, so it is Legendrian. Furthermore, the associated contact structure is isotopic to the one we started with by Gray stability.

□

Remark 4.3. In dimension 3, every curve in a page is Lagrangian, but to realize a curve as a Legendrian one needs to perturb transversely to a page. Hence we cannot directly formulate an analogue to Lemma 4.2. On the other hand, in dimension 3 one can always find a supporting open book such that a Legendrian lies in a page, see [4], section 4, for a discussion of the 3-dimensional situation.

Given the Lagrangian sphere L_S , we get a compactly supported symplectomorphism τ_{L_S} , a right-handed Dehn twist along L_S . We can now change the monodromy of the contact open book by adding Dehn twists along L_S , but we can also perform contact surgery along L_S . We shall now show that these operations coincide.

4.2.1. Surgery and monodromy. The goal of this section is to provide a proof of the following folk theorem. This result is well known in dimension 3, [4].

Theorem 4.4. *Let $\text{Open}(\Sigma, \psi)$ be a contact open book with Legendrian sphere L_S , that restricts to a Lagrangian sphere in Σ . Denote the contact manifold obtained from $\text{Open}(\Sigma, \psi)$ by Legendrian surgery along L_S by $\widetilde{\text{Open}(\Sigma, \psi)}_{L_S}$. Then the contact manifolds*

$$\text{Open}(\Sigma, \psi \circ \tau_{L_S}) \cong \widetilde{\text{Open}(\Sigma, \psi)}_{L_S}$$

are contactomorphic.

Proof. The proof has two steps. In the first step we shall show that Legendrian surgery on $\text{Open}(\Sigma, \psi)$ along L_S yields a contact manifold with a supporting open book $(B, \hat{\vartheta}_r)$, where B is the binding and $\hat{\vartheta}_r$ the map to S^1 . The new supporting open book has the same page as the supporting open book before the surgery, and we can localize the monodromy. In the second step we determine how the monodromy is changed.

Step 1: Supporting open book after surgery

The contact open book $\text{Open}(\Sigma, \psi)$ gives rise to a contact manifold (M, ξ) with a supporting open book (B, ϑ) , where B is a codimension 2 submanifold of M and $\vartheta : M - B \rightarrow S^1$ a fiber bundle over S^1 . We think of L_S as a Legendrian submanifold of (M, ξ) lying in the page $0 \in \mathbb{R}/\mathbb{Z} \cong S^1$. Choose a neighborhood $\nu(L_S)$ of L_S such that $\nu(L_S)$ is contactomorphic to $T^*L_S \times \mathbb{R}$, and such that $\nu(L_S) \subset \vartheta^{-1}(]-\varepsilon, \varepsilon[)$. In particular, we can restrict the map ϑ to a map $\vartheta_r := \vartheta|_{\nu(L_S)} : \nu(L_S) \rightarrow]-\varepsilon, \varepsilon[$.

Note that $\nu(L_S)$ can be identified with a neighborhood of $\{z = 0\}$ in the hypersurface of contact type $\{w^2 = 1\} \subset \mathbb{C}^n$, as described before in the interlude on the “flat” Weinstein model. Let us use the identification to choose a specific model for ϑ_r .

By isotoping the open book and Gray stability we can assume that the restricted map ϑ_r has the form

$$\begin{aligned} \vartheta_r : \nu(L_S) &\longrightarrow]-\varepsilon, \varepsilon[\\ (z, w) &\longmapsto z \cdot w \end{aligned}$$

Indeed, since the Reeb field on $\{w^2 = 1\}$ is given by

$$R_B = w \partial_z,$$

we see that $R(\vartheta_r) > 0$, so ϑ_r gives also a supporting open book for (M, ξ) .

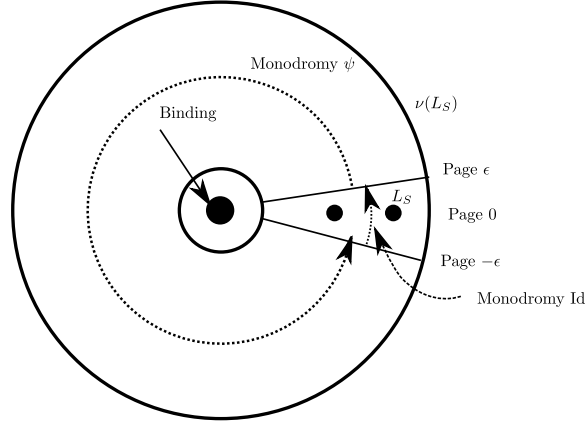


FIGURE 7. Pages around a Legendrian sphere

Next perform Legendrian surgery along L_S using the “flat” Weinstein model as described in Section 3.1. This means that we remove a neighborhood $\nu_{M,\varepsilon}(L_S)$ of L_S and glue in the set S_1 , which is the zero set of the function $F(z; w) = -f(w^2) + g(z^2)$. In our setup, the Reeb field on S_1 is a positive multiple of the Hamiltonian vector field of F , given by

$$X_F = \frac{\partial F}{\partial z} \partial_w - \frac{\partial F}{\partial w} \partial_z = 2g'z \partial_w + 2f'w \partial_z.$$

By our choice of the functions f and g we see that

$$R(z \cdot w) = NX_F(z \cdot w) = N(2g'|z|^2 + 2f'|w|^2) > 0,$$

so the function ϑ_r also defines a suitable open book projection on S_1 .

Step 2: Monodromy

Let us now investigate the monodromy. The change of the monodromy can be localized in an ε -neighborhood of page 0, and furthermore this change of monodromy does not depend on the choice of L_S , since we have described the entire setup with the Weinstein model. So we see that

$$\widetilde{\text{Open}(\Sigma, \psi)}_{L_S} \cong \text{Open}(\Sigma, \psi \circ \psi_{L_S}),$$

where ψ_{L_S} is the change in monodromy. Hence we only need to see what Legendrian surgery does to the monodromy in a single model situation to determine ψ_{L_S} .

Monodromy before surgery Let us first compute the monodromy from page $-\varepsilon$ to page ε before the surgery. Take a point $(z_{-\varepsilon}, w_{-\varepsilon})$ in A , i.e. $|w_{-\varepsilon}|^2 = 1$. Since we start at page $-\varepsilon$, we have $\vartheta(z_{-\varepsilon}, w_{-\varepsilon}) = -\varepsilon$, so we may write

$$z_{-\varepsilon} = -\varepsilon w_{-\varepsilon} + r_{-\varepsilon},$$

where $r_{-\varepsilon} \cdot w_{-\varepsilon} = 0$. In particular $(w_{-\varepsilon}, r_{-\varepsilon})$ can be seen as points in T^*S^n .

The Reeb flow is the reparametrized Hamiltonian flow for the Hamiltonian $w^2 = 1$. Hence we have the following solutions to the flow equations:

$$z(s) = (s - \varepsilon)w_{-\varepsilon} + r_{-\varepsilon}, \quad w(s) = w_{-\varepsilon}.$$

We see that the Reeb flow transports the point $(z_{-\varepsilon}, w_{-\varepsilon})$ after time $s = 2\varepsilon$ to page $+\varepsilon$. Since the decomposition $z = sw + r$ is preserved, we conclude that the monodromy is trivial.

Monodromy after surgery Let us now consider the monodromy after surgery. We need to compare with the monodromy before the surgery and for that we shall use the following map to identify subsets of S_{-1} with subsets of S_1 . Take a point $(z_{-\varepsilon}, w_{-\varepsilon})$ with $|w_{-\varepsilon}|^2 = 1$, but $z_{-\varepsilon} \neq 0$. This point can be seen as a point in S_{-1} . For $a > 0$ we define the Liouville vector field

$$X_a := (1 + a)z\partial_z - aw\partial_w.$$

Later computations will be simpler when we take the limit $a \rightarrow \infty$. For now we shall use the vector field X_a though for some fixed a .

Let us now identify S_{-1} with the flat parts of S_1 , i.e. those subsets of S_1 where either f' or g' is zero. The time- t flow of X_a sends $(z_{-\varepsilon}, w_{-\varepsilon})$ to

$$(e^{(1+a)t}z_{-\varepsilon}, e^{-at}w_{-\varepsilon}).$$

So if we send $(z_{-\varepsilon}, w_{-\varepsilon})$ to a flat piece of S_1 we obtain

$$(z'_{-\varepsilon}, w_{-\varepsilon}) = \left(\frac{1}{|z_{-\varepsilon}|} z_{-\varepsilon}, |z_{-\varepsilon}|^{-\frac{a}{1+a}} w_{-\varepsilon} \right).$$

Let us now consider the limit $a \rightarrow \infty$ to enable explicit computations. We can then later argue using isotopies that our final answer remains valid. The points on the flat piece are now

$$(z'_{-\varepsilon}, w'_{-\varepsilon}) = \left(\frac{1}{|z_{-\varepsilon}|} z_{-\varepsilon}, |z_{-\varepsilon}| w_{-\varepsilon} \right).$$

Let us now flow with the Reeb vector field to page ε . As stated before, the Reeb vector field is a reparametrization of the Hamiltonian vector field

$$X_F = 2g'z\partial_w + 2f'w\partial_z.$$

On the flat piece of S_1 we are interested in, we have

$$X_F = 2z\partial_w.$$

The time- s flow sends the point $(z'_{-\varepsilon}, w'_{-\varepsilon})$ to

$$(z'(s), w'(s)) = (z'_{-\varepsilon}, w'_{-\varepsilon} + 2sz'_{-\varepsilon}) = \left(\frac{1}{|z_{-\varepsilon}|} z_{-\varepsilon}, |z_{-\varepsilon}| w_{-\varepsilon} + 2s \frac{1}{|z_{-\varepsilon}|} z_{-\varepsilon} \right).$$

The page number can be computed as

$$\vartheta(z'(s), w'(s)) = -\varepsilon + 2s.$$

Hence we see that at page ε we have arrived at point

$$(z'_\varepsilon, w'_\varepsilon) = \left(\frac{1}{|z_{-\varepsilon}|} z_{-\varepsilon}, |z_{-\varepsilon}| w_{-\varepsilon} + 2\varepsilon \frac{1}{|z_{-\varepsilon}|} z_{-\varepsilon} \right).$$

We transform this point back to S_{-1} to compare with the monodromy before surgery. We find

$$(z_\varepsilon, w_\varepsilon) = \left(|w'_\varepsilon| z'_\varepsilon, \frac{1}{|w'_\varepsilon|} w'_\varepsilon \right).$$

Note that $|w'_\varepsilon| = \sqrt{|z_{-\varepsilon}|^2 + 4\varepsilon z_{-\varepsilon} \cdot w_{-\varepsilon} + 4\varepsilon^2} = |z_{-\varepsilon}|$, so we see that the monodromy corresponding to Legendrian surgery is given by

$$(z_\varepsilon, w_\varepsilon) = \left(z_{-\varepsilon}, w_{-\varepsilon} + 2\varepsilon \frac{z_{-\varepsilon}}{|z_{-\varepsilon}|^2} \right).$$

Recognizing the monodromy as a Dehntwist To interpret the monodromy as a Dehn twist, recall the contactomorphism from Section 3.1

$$\begin{aligned}\psi_W : \mathbb{R} \times T^*S^n &\longrightarrow S_{-1} \\ (z; q, p) &\longmapsto (zq + p, q),\end{aligned}$$

where we regard T^*S^n as a subspace of \mathbb{R}^{2n+2} using coordinates (q, p) subject to the relations $q^2 = 1$ and $q \cdot p = 0$.

Using this description, we decompose $(z_\varepsilon, w_\varepsilon)$ as

$$z_\varepsilon = \varepsilon w_\varepsilon + r_\varepsilon,$$

where $|w_\varepsilon|^2 = 1$ and $w_\varepsilon \cdot r_\varepsilon = 0$. We have a similar decomposition for the initial point,

$$z_{-\varepsilon} = -\varepsilon w_{-\varepsilon} + r_{-\varepsilon}.$$

Let us now compute the $(w_\varepsilon, r_\varepsilon) \in T^*S^n$ in terms of $(w_{-\varepsilon}, r_{-\varepsilon})$ to compare with Dehn twists. We have

$$w_\varepsilon = w_{-\varepsilon} + 2\varepsilon \frac{-\varepsilon w_{-\varepsilon} + r_{-\varepsilon}}{r_{-\varepsilon}^2 + \varepsilon^2} = \frac{r_{-\varepsilon}^2 - \varepsilon^2}{r_{-\varepsilon}^2 + \varepsilon^2} w_{-\varepsilon} + \frac{2\varepsilon}{r_{-\varepsilon}^2 + \varepsilon^2} r_{-\varepsilon}.$$

and

$$\begin{aligned}r_\varepsilon &= z_\varepsilon - \varepsilon w_\varepsilon = -\varepsilon w_{-\varepsilon} + r_{-\varepsilon} - \varepsilon w_{-\varepsilon} - \frac{2\varepsilon^2 r_{-\varepsilon}}{r_{-\varepsilon}^2 + \varepsilon^2} + \frac{2\varepsilon^3 w_{-\varepsilon}}{r_{-\varepsilon}^2 + \varepsilon^2} \\ &= -\frac{2\varepsilon r_{-\varepsilon}^2}{r_{-\varepsilon}^2 + \varepsilon^2} w_{-\varepsilon} + \frac{r_{-\varepsilon}^2 - \varepsilon^2}{r_{-\varepsilon}^2 + \varepsilon^2} r_{-\varepsilon}\end{aligned}$$

Now observe that the functions $\frac{\varepsilon^2 - r_{-\varepsilon}^2}{r_{-\varepsilon}^2 + \varepsilon^2}$ and $\frac{2\varepsilon |r_{-\varepsilon}|}{r_{-\varepsilon}^2 + \varepsilon^2}$ are the standard functions for the rational parametrization of the circle, i.e. we can regard them as $\cos g(r_\varepsilon)$ and $\sin g(r_\varepsilon)$ of some increasing function $g(r_\varepsilon)$, respectively. With this in mind, we see that

$$\begin{pmatrix} w_\varepsilon \\ r_\varepsilon \end{pmatrix} = \begin{pmatrix} -\cos g(r_\varepsilon) & \sin g(r_\varepsilon)/|r_{-\varepsilon}| \\ -\sin g(r_\varepsilon)|r_{-\varepsilon}| & -\cos g(r_\varepsilon) \end{pmatrix} \begin{pmatrix} w_{-\varepsilon} \\ r_{-\varepsilon} \end{pmatrix}.$$

Note that we can recognize this map as a *right-handed* Dehn twist $\sigma_{\tilde{g}(|p|)}$ if we define $\tilde{g}(x) := \pi - g(x)$.

Isotopy to correct the map Note that we have made two approximations in the above computation.

- We have ignored the rounding piece of size δ in S_1 .
- We have used X_∞ rather than X_a for some $a \in \mathbb{R}$.

Let us argue that these approximations can be corrected for by performing isotopies with compact support on T^*S^n . Indeed, note the following

- Fix the smoothing parameter $\delta > 0$ in S_1 . Observe that we miss the monodromy of points (z, w) such that

$$1 - \delta < |z|^2, |w|^2 < 1 + \delta.$$

Note that we can bound the effect of the flow of X_F by $C\delta$ for these points. More precisely, if we follow the above scheme, but send points in S_{-1} to S_1 rather than just into the flat pieces of S_{-1} , then we have the following bounds:

$$|w_\varepsilon - w_{-\varepsilon}| < C\delta, \quad |r_\varepsilon - r_{-\varepsilon}| < C\delta.$$

In particular, we can choose δ small enough to see that the correction in the round piece can be isotoped to the identity by an isotopy with compact support in T^*S^n .

- By choosing a sufficiently large, the map obtained from the above procedure with the Liouville vector field X_a rather than X_∞ is arbitrarily close to the one obtained with X_∞ . Hence we also find an isotopy to correct for this.

It follows that the monodromy obtained by the surgery is indeed a right-handed Dehn twist. \square

An immediate corollary of the above theorem is the following well-known statement about the relation between fillability and the monodromy of an open book.

Corollary 4.5. *Let $M := \text{Open}(\Sigma, \psi)$ be a contact open book with a Stein page Σ and a monodromy that is the product of right-handed Dehn twists. Then M is Stein fillable.*

Proof. The contact manifold $\tilde{M} := \text{Open}(\Sigma, \text{Id})$ is Stein-fillable with filling $\tilde{W} := \Sigma \times D^2$. By Theorem 4.4, we see that we obtain M from \tilde{M} by critical surgery along Legendrian spheres. Since this can also be done on cobordism level, we obtain a Stein filling for M by attaching critical handles along Legendrian spheres to \tilde{W} . \square

Remark 4.6. The actual Stein filling depends on the precise factorization of the monodromy into right-handed Dehn twists. Different factorizations of a given monodromy can give rise to distinct Stein fillings for the same contact manifold which is determined by the monodromy itself rather than its factorization into Dehn twists.

4.3. Stabilization. Let us now consider the contact open book $M = \text{Open}(\Sigma^{2n}, \psi)$ and suppose that L is an embedded Lagrangian disk D^n in the page Σ whose boundary ∂L is a Legendrian sphere in $\partial\Sigma$.

As in the previous section we define a new contact open book by

$$\tilde{M} = \text{Open}(\tilde{\Sigma}, \tilde{\psi}),$$

where $\tilde{\Sigma}$ is obtained from Σ by n -handle attachment along ∂L . The monodromy $\tilde{\psi}$ restricts to the identity on the attached n -handle and coincides with ψ on Σ . Note that $\tilde{\Sigma}$ contains a Lagrangian sphere L_S spanned by the Lagrangian disk L and the core of the n -handle.

Remark 4.7. Let us interpret the critical attachment of a handle h to the page Σ of M as subcritical handle attachment to the symplectic cobordism $W := [0, 1] \times M$ as in Proposition 4.1. Denote the result of this handle attachment by \tilde{W} .

We see that $L_S \subset \tilde{W}$ intersects the belt sphere of the attached handle transversely in one point, since the part of L_S in the handle has the form $\text{core} \times \{p\} \subset h \times D^2$.

Definition 4.8. Let $M = \text{Open}(\Sigma, \psi)$ be a contact open book with an embedded Lagrangian disk L as above. The contact open book

$$\tilde{M} := \text{Open}(\tilde{\Sigma}, \tilde{\psi} \circ \tau_{L_S})$$

is called the **stabilization** of $\text{Open}(\Sigma, \psi)$ along L .

The following claim is a well-known statement due to Giroux.

Proposition 4.9 (Giroux). *The stabilization of a contact open book $\text{Open}(\Sigma, \psi)$ along a Lagrangian disk L bounding a Legendrian sphere in $\partial\Sigma$ is contactomorphic to the contact manifold $\text{Open}(\Sigma, \psi)$.*

Proof. Let $M^{2n+1} = \text{Open}(\Sigma, \psi)$ be a contact open book and L a Lagrangian disk D^n in a page Σ . To stabilize the open book, we first need to attach an n -handle to Σ along ∂L to obtain a new page $\tilde{\Sigma}$. The submanifold ∂L of the binding corresponds to an isotropic sphere S , so on the level of contact manifolds, the first step of stabilizing is realized by performing contact surgery along the isotropic sphere S as in Proposition 4.1. In the language of symplectic cobordisms, we start with a compact piece of the symplectization of M , say $[0, 1] \times M$, and then attach an n -handle to $\{1\} \times M$ along $\{1\} \times S$.

The next step of the stabilization consists of changing the monodromy; we have a Lagrangian sphere in the new page $\tilde{\Sigma}$, which we denote by L . Note that we can assume that L is also a Legendrian sphere in $\text{Open}(\tilde{\Sigma}, \tilde{\psi})$ by Lemma 4.2. The stabilization is given by

$$M_S = \text{Open}(\tilde{\Sigma}, \psi \circ \tau_L).$$

On the level of contact manifolds, this change of monodromy can be realized by performing Legendrian surgery along L , as we can apply Theorem 4.4. In cobordism language, this amounts to attaching an $(n+1)$ -handle to W along the Legendrian sphere L , as described in Section 3.

By construction of the stabilization, the Legendrian sphere L intersects the belt sphere of the previously attached n -handle in precisely one point, see Remark 4.7. This means that the interpretation of the stabilization procedure in terms of symplectic handle attachment fits exactly the description of symplectic handle cancellation in Section 3.2. Hence we apply the handle cancellation lemma 3.11 to see that the stabilization yields a contactomorphic manifold. \square

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